

A homotopical algebra of graphs related to zeta series.

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Abstract: The purpose of this paper is to develop a homotopical algebra for graphs, relevant to zeta series and spectra of finite graphs. More precisely, we define a Quillen model structure in a category of graphs (directed and possibly infinite, with loops and multiple arcs allowed). The weak equivalences for this model structure are the Acyclics (graph morphisms which preserve cycles). The cofibrations and fibrations for the model are determined from the class of Whiskerings (graph morphisms produced by grafting trees). Our model structure seems to fit well with the importance of acyclic directed graphs in many applications.

In addition to the weak factorization systems which form this model structure, we also describe two Freyd-Kelly factorization systems based on Folding, Injecting, and Covering graph morphisms.

0. Introduction.

In this paper we develop a notion of homotopy within graphs, and demonstrate its relevance to the study of zeta series and spectrum of a finite graph. We will work throughout with a particular category of graphs, described in Section 1 below. Our graphs will be directed and possibly infinite, with loops and multiple arcs allowed.

Let us explain what we mean by homotopy here.

We are not concerned with the geometric realization of graphs as one-dimensional topological spaces. Since one-dimensional CW complexes are homotopic to disjoint unions of joins of circles, the usual invariants from algebraic topology cannot see much of the structure of a graph in this way. In any case, *directed* graphs are definitely not just part of topology (they are perhaps more related to new areas of *directed topology*, as in Fajstrup and Rosický [2007]).

Homotopy originally referred to topological deformation of structure. But Quillen's remarkable notes on homotopical algebra [1967] gave abstract axioms for working with concepts of homotopy in rather general categories. When these axioms are satisfied in a category, we say that we have given a "model structure" there. Quillen's axioms have led to new insights and developments in settings such as chain complexes and homological algebra, simplicial sets, topos theory, and small categories (including monoids, groups, groupoids, and posets). References include Thomason [1980], Joyal and Tierney [1991], Dwyer and Spalinski [1995], Cisinski [2002], and many others. Also, recent proofs of the Bloch-Kato and Milnor conjectures are based upon development of a homotopical algebra for schemes; see Voevodsky and Morel [1999].

A central part of giving a model structure in a category is the specification of which morphisms in the category are to be called "weak equivalences". In most applications, the weak equivalences are defined to be those morphisms which preserve some interesting invariant, such as homotopy type for topology, homology for chain complexes, geometric realization for simplicial sets, and nerve or topos of presheafs for small categories.

In our model structure we use the cycle structure of directed graphs to determine our weak equivalences. More precisely, we take as our weak equivalences the "acyclic" graph morphisms, which neither create nor destroy cycles. We hope that our model structure fits well with the role that acyclic directed graphs play in in applications such as computer algorithms, analysis of the internet, random walks and markov chains, and representations of quivers.

In section 1 we set up our category \mathbf{Gph} of graphs. In section 2 we give background on weak factorization systems in general, and establish an example with classes of graph morphisms which we call Whiskerings and Surjectings. In section 3, which is somewhat of a digression, we discuss Freyd-Kelly factorization systems in general, and consider two examples in \mathbf{Gph} , involving classes of graph morphisms which we call Injectings, Foldings, and Coverings. In section 4 we give axioms for model structures in general, and define our model structure on \mathbf{Gph} . In section 5 we associate to each finite directed graph X a zeta series $Z_X(u)$. We show that if $f : X \rightarrow Y$ is an acyclic graph morphism, then $Z_X(u) = Z_Y(u)$ and the eigenvalues of the adjacency matrices of X and Y agree "up to zero eigenvalues". The paper ends with two appendices.

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1. A category of graphs.

Let us establish precisely the objects and morphisms for our category \mathbf{Gph} of graphs. For our purposes, a *graph* is a data structure $X = (X_0, X_1, s, t)$ with a set X_0 of *nodes*, a set X_1 of *arcs*, and a pair of functions $s, t : X_1 \rightarrow X_0$ which specify the *source* and *target* nodes of each arc. We may say that $a \in X_1$ is an arc *from node $s(a)$ to node $t(a)$* . A *graph morphism* $f : X \rightarrow Y$ is a pair of functions $f_1 : X_1 \rightarrow Y_1$ and $f_0 : X_0 \rightarrow Y_0$ such that $s \circ f_1 = f_0 \circ s$ and $t \circ f_1 = f_0 \circ t$.

Let \mathbf{N} denote the graph with one node and no arcs. Let \mathbf{A} denote the graph with one arc and two nodes (its source and target). Then the set of nodes of a graph X can be identified with the set of graph morphisms from \mathbf{N} to X , and the set of arcs of X can be identified with the set of graph morphisms from \mathbf{A} to X .

There is a stimulating discussion in Lawvere [1989] of \mathbf{Gph} as the category of presheafs on the small category with objects 0 and 1 and two non-identity morphisms from 0 to 1. It follows that \mathbf{Gph} is a topos, and thus a category with many nice geometric and algebraic and logical properties; see Mac Lane and Moerdijk [1994], for instance. We just call attention to a few aspects here.

The category \mathbf{Gph} has all products, and all coproducts (sums); it also has pull-backs (fiber products) and pushouts. Also, products distribute over coproducts, etc. As in any presheaf category, these categorical constructions are performed “elementwise”, where a graph has two types of elements, the nodes and the arcs. For instance, the empty product (terminal object) 1 is the graph with one node and one arc (which is a loop); and the empty coproduct (initial object) 0 is the graph with no nodes and no arcs.

The category \mathbf{Gph} has geometric aspects; for instance, it is a cartesian closed category like a good category of “spaces”. The category \mathbf{Gph} also has logical aspects; for instance, there is a graph Ω which acts as generalized truth-values for graphs, in that graph morphisms $X \rightarrow \Omega$ classify sub-graphs of X (see Session 32 in Lawvere and Schanuel [1997]).

2. Two classes of graph morphisms, and a weak factorization system.

The *path graph* \mathbf{P}_n has nodes $\{i : 0 \leq i \leq n\}$ and arcs $\{(i-1, i) : 1 \leq i \leq n\}$, with $s((i-1, i)) = i-1$ and $t((i-1, i)) = i$. Note that $\mathbf{P}_0 = \mathbf{N}$ and $\mathbf{P}_1 = \mathbf{A}$. A *path of length n* in a graph X is just a graph morphism $\alpha : \mathbf{P}_n \rightarrow X$; we define $s(\alpha) = \alpha(0)$ and $t(\alpha) = \alpha(n)$. If a is an arc in X such that $t(a) = s(a)$, then we define $\alpha a : \mathbf{P}_{n+1} \rightarrow X$, the *concatenation of α and a* .

Let us introduce some useful shorthand for arcs in a graph X . For any node x in a graph X , let $X(x, *)$ denote the set of those arcs in X which have source x , and let $X(*, x)$ denote the set of arcs with target x . Note that a graph morphism $f : X \rightarrow Y$ induces a function $f : X(x, *) \rightarrow Y(f(x), *)$, etc.

Here is our first class of graph morphisms.

Definition: A graph morphism $f : X \rightarrow Y$ is *Surjecting* if $f : X(x, *) \rightarrow Y(f(x), *)$ is a surjective function for all $x \in X_0$.

A *discrete graph* is one with no arcs. We say that a node x is a *root* of the graph X if $X(*, x)$ is empty. Let $R(X)$ denote the set of roots in X , viewed as a discrete subgraph of X . A *rooted tree* is a graph T with one root r such that, for each node x in T , there is a unique (directed) path in T from r to x . For example, the path graph \mathbf{P}_n is a rooted tree; and so is the infinite path \mathbf{P}_∞ whose nodes are the set of non-negative integers and whose arcs are the set of ordered-pairs $(i-1, i)$ of non-negative integers, with $s((i-1, i)) = i-1$ and $t((i-1, i)) = i$.

For any node x in a graph X , we can define a rooted tree $T_x X$, the *tree of paths in X leaving x* . The nodes in $T_x X$ are the finite paths in X with source x (note that x is considered as a path of length 0 in X); the arcs in $T_x X$ are the triples $(\alpha, a, \alpha a)$ where αa is the concatenation of path α and arc a in X ; and $s(\alpha) = s(\alpha, a, \alpha a)$ and $t(\alpha, a, \alpha a) = t(a)$. There are natural graph morphisms $T_x X \rightarrow X$ given by $\alpha \mapsto t(\alpha)$ and $(\alpha, a, \alpha a) \mapsto a$.

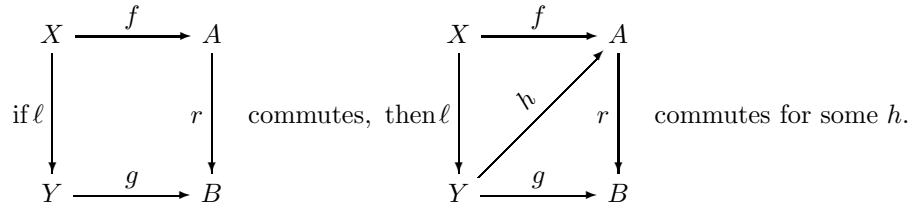
A *rooted forest* F is a coproduct of rooted trees. If F is a forest with roots $R(F)$, then we may form a new graph X_F as the pushout of the graph morphisms $r : R(F) \rightarrow F$ and $f : R(F) \rightarrow X$; we say that X_F is *formed by attaching the forest F to X along R* . For instance, if r is a root in a tree T and x is any node in a graph X , then X_T is formed by attaching the tree T to X at the node x (along the graph morphisms $r : \mathbf{N} \rightarrow T$ and $x : \mathbf{N} \rightarrow X$).

Definition: A graph morphism $f : X \rightarrow Y$ is a *Whiskering* if Y is formed by attaching some rooted forest to X .

For example, the graph morphism $\mathbf{s} : \mathbf{N} \rightarrow \mathbf{A}$ is a Whiskering, where \mathbf{s} exhibits the node graph \mathbf{N} as the source subgraph of the arc graph \mathbf{A} . Also, every isomorphism is a Whiskering, and $R(F) \rightarrow F$ is a Whiskering if F is a rooted forest.

Our goal in this section is to demonstrate some remarkable factorization properties of Surjectings and Whiskerings. Here is the conceptual background.

Definition: Let $\ell : X \rightarrow Y$ and $r : A \rightarrow B$ be morphisms in a category \mathcal{S} . We say that $\ell \dagger r$ when, for all f and g ,



We say that h is a *filler* for the commutative diagram. We may also say that h *lifts* g along r , or that h *drops* (“extends”) f along ℓ . Given two classes \mathcal{L} and \mathcal{R} of morphisms, we say $\mathcal{L} \dagger \mathcal{R}$ when we have $\ell \dagger r$ for every $\ell \in \mathcal{L}$ and every $r \in \mathcal{R}$. Given a class \mathcal{F} of morphisms we may define

$$\mathcal{F}^\dagger = \{r : f \dagger r, \forall f \in \mathcal{F}\} \quad \text{and} \quad {}^\dagger\mathcal{F} = \{\ell : \ell \dagger f, \forall f \in \mathcal{F}\}.$$

Definition: A *weak factorization system* in \mathcal{S} is given by two classes \mathcal{L} and \mathcal{R} such that $\mathcal{L}^\dagger = \mathcal{R}$ and $\mathcal{L} = {}^\dagger\mathcal{R}$ and such that, for any morphism c in \mathcal{S} , there exist $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$ with $c = r \circ \ell$.

The notion of weak factorization system has become a part of homotopical algebra; see Section 4 here. The appendix on Galois connections also provides some context.

The following three propositions combine to show that Surjectings and Whiskerings give a weak factorization system in Gph. Our Proposition 1 was inspired by an argument in Enochs and Herzog [1999].

Proposition 1. Any graph morphism $f : X \rightarrow Y$ factors as a Whiskering followed by a Surjecting.

Proof: Recall that for each node y in Y we have the tree $T_y Y$ of paths in Y leaving y . From f we construct the rooted forest $F = \sum_{x \in X_0} T_{f(x)} Y$, with roots $R = X_0$ considered as discrete subgraph of X . The pushout of $R \rightarrow F$ along the subgraph inclusion $R \rightarrow X$ defines a Whiskering $w : X \rightarrow X_F$. We have $g : F \rightarrow Y$ as coproduct of the morphisms $T_{f(x)} Y \rightarrow Y$, and since $f : X \rightarrow Y$ and $g : F \rightarrow Y$ agree on R , they determine a unique graph morphism $p : X_F \rightarrow Y$. Note that $f = p \circ w$.

Let us show that p is a Surjecting. For any node z in X_F we must show that $p : X_F(z, *) \rightarrow Y(p(z), *)$ is surjective. But z is either a node x or a path in Y with source $f(x)$, for some $x \in X_0$.

In the first case, we have $p : X_F(x, *) \rightarrow Y(f(x), *)$ with $X_F(x, *) = X(x, *) \cup T_y Y(f(x), *)$, and $T_y Y(f(x), *) \rightarrow Y(f(x), *)$ is a bijection.

In the second case, we have $p : X_F(\alpha, *) \rightarrow Y(p(\alpha), *)$ with $X_F(\alpha, *) = T_{f(x)} Y(\alpha, *) = Y(t(\alpha), *) = Y(p(\alpha), *)$.

In either case, $p : X_F(z, *) \rightarrow Y(p(z), *)$ is surjective. QED

Proposition 2. Whiskering \dagger Surjecting.

Proof: Let $f : Z \rightarrow Y$ be Surjecting. First we show lifting of rooted trees. More precisely, if T is a rooted tree with root x and we have the following commutative diagram

$$\begin{array}{ccc} \mathbf{N} & \xrightarrow{z} & Z \\ \downarrow x & & \downarrow f \\ T & \xrightarrow{g} & Y \end{array}$$

then there is a filler $h : T \rightarrow Z$. This follows by induction on the length of path from root to nodes of T , as follows. Suppose that we have extended h to paths of length n and let αa be a path of length $n + 1$ in T . Let $x' = t(\alpha)$. Then $f(h(x')) = g(x')$ and $g(a) \in Y(g(x'), *)$ and $f : Z(h(x'), *) \rightarrow Y(g(x'), *)$ is a surjective function, so there exists an arc $a' \in Z(h(x'), *)$ so that $f(a') = g(a)$. We extend h to αa by $h(a) = a'$.

More generally, consider any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g'} & Z \\ \downarrow w & & \downarrow f \\ X_F & \xrightarrow{g} & Y \end{array} \quad \text{with } w \text{ a Whiskering given by} \quad \begin{array}{ccc} R(F) & \xrightarrow{i} & X \\ \downarrow & & \downarrow w \\ F & \xrightarrow{j} & X_F \end{array}$$

We want to define a filler $h : X_F \rightarrow Z$, by extending g' along every tree T in F . This is possible since the square is commutative and f is Surjecting. QED

Proposition 3. (Whiskering, Surjecting) is a weak factorization system.

Proof: By the preceding proposition we have $\text{Whiskering} \subseteq {}^\dagger\text{Surjecting}$ and $\text{Surjecting} \subseteq \text{Whiskering}^\dagger$.

If $f : X \rightarrow Y$ is not in Surjecting then there exists some $x \in X_0$ and some $a \in Y(f(x), *)$ which is not in the image of $X(x, *) \rightarrow Y(f(x), *)$. Consider the Whiskering $\mathbf{s} : \mathbf{N} \rightarrow \mathbf{A}$ and the commutative diagram

$$\begin{array}{ccc} \mathbf{N} & \xrightarrow{x} & X \\ \downarrow \mathbf{s} & & \downarrow f \\ \mathbf{A} & \xrightarrow{a} & Y \end{array}$$

for which there is no filler. This shows that $f \notin \text{Surjecting}$ implies $f \notin \text{Whiskering}^\dagger$. It follows that $\text{Whiskering}^\dagger = \text{Surjecting}$. We will show that $f \notin \text{Whiskering}$ implies $f \notin {}^\dagger\text{Surjecting}$, by factoring. Suppose that $f : X \rightarrow Y$ with $f \notin \text{Whiskering}$; then $f = p \circ w$ with some Whiskering $w : X \rightarrow X_F$ and some Surjecting $p : X_F \rightarrow Y$. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{w} & X_F \\ \downarrow f & & \downarrow p \\ Y & \xrightarrow{\text{id}} & Y \end{array}$$

If this had a filler $h : Y \rightarrow X_F$, then we would have $p \circ h = \text{id}$. This would exhibit f as a “morphism retract” of w (a retract in the morphism category). But we show in the next lemma that this would give us the desired contradiction, finishing our proof.

Lemma. Whiskerings are stable with respect to morphism retract.

Proof: First we show that any retract of a rooted tree is a rooted tree. If T is a rooted tree with root x_0 and we have the following commutative diagram with $r \circ s = \text{id}_{T'}$

$$\begin{array}{ccccc} \mathbf{N} & \xrightarrow{\text{id}} & \mathbf{N} & \xrightarrow{\text{id}} & \mathbf{N} \\ \downarrow x & & \downarrow x_0 & & \downarrow x \\ T' & \xrightarrow{s} & T & \xrightarrow{r} & T' \end{array}$$

then T' is a rooted tree with root x . This is clear since, for any node x' in T' , the unique path α from x_0 to $s(x')$ gives $r \circ \alpha$ a path from x to $r(s(x')) = x'$, and there can be no other path in T' from x to x' .

More generally, consider any commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{s'} & Z & \xrightarrow{r'} & X \\ \downarrow f & & \downarrow w & & \downarrow f \\ Y & \xrightarrow{s} & Z_F & \xrightarrow{r} & Y \end{array}$$

such that w is a Whiskering, $r \circ s = \text{id}_Y$ and $r' \circ s' = \text{id}_X$. The fact that $r \circ s = \text{id}_Y$ implies that s is injective on nodes and arcs. Also w is injective on nodes and arcs since it is a Whiskering. This implies that f is injective on nodes and arcs, since the first square is commutative.

We will describe a rooted forest F' with roots R' a discrete subgraph of X , such that $Y = X_{F'}$.

The Whiskering w is given by a rooted forest F whose R form a discrete subgraph of Z . Let $R' = R \cap X$ as subgraphs of Z . Then R' is a discrete subgraph of X . For each $x \in R'$, consider the tree T in the forest F with root $x_0 = s(x)$. Then $r(T)$ is a retract of T , so $r(T)$ is a tree with root $x = r(x_0)$. Let $F' = \sum_{x \in R'} r(T)$. Then $Y = X_{F'}$, and we are done. QED

3. Two Freyd-Kelly factorization systems in graphs.

The notion of weak factorization system discussed in the preceding section is a “weakened” version of an older notion in category theory.

Definition: For morphisms ℓ and r in a category \mathcal{S} , we say that $\ell \perp r$ (ℓ is *orthogonal* to r) when

$$\begin{array}{ccc} \begin{array}{ccc} X & \xrightarrow{f} & A \\ \downarrow \ell & & \downarrow r \\ Y & \xrightarrow{g} & B \end{array} & \text{commutes, then} & \begin{array}{ccc} X & \xrightarrow{f} & A \\ \downarrow \ell & \nearrow h & \downarrow r \\ Y & \xrightarrow{g} & B \end{array} \\ \text{if } \ell & & \text{commutes for unique } h. \end{array}$$

Thus $\ell \perp r$ means that every equation $rf = g\ell$ has a unique “solution” h with $f = h\ell$ and $g = rh$, so that the original equation is just $r(h\ell) = (rh)\ell$.

Given a class \mathcal{F} of morphisms in \mathcal{S} we define

$$\mathcal{F}^\perp = \{r : \mathcal{F} \perp r\} \quad \text{and} \quad {}^\perp\mathcal{F} = \{\ell : \ell \perp \mathcal{F}\} \quad \text{and} \quad {}^\perp(\mathcal{F}^\perp) \quad \text{and} \quad ({}^\perp\mathcal{F})^\perp.$$

Definition: A *Freyd-Kelly factorization system* in \mathcal{S} is given by two classes \mathcal{L} and \mathcal{R} with $\mathcal{L}^\perp = \mathcal{R}$ and $\mathcal{L} = {}^\perp\mathcal{R}$, such that for any morphism c in \mathcal{S} , there exist $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$ with $c = r \circ \ell$.

This notion of factorization system is given in section 2 of Freyd and Kelly [1972]. A Freyd-Kelly factorization system is often just called a *factorization system*. We may use notation $((\mathcal{L}, \mathcal{R}))$ to indicate a Freyd-Kelly factorization system.

A basic example is the epimorphic, monomorphic factorization system $((\mathcal{E}, \mathcal{M}))$ in the category of sets: $\{2 \rightarrow 1\}^\perp = \mathcal{M}$ is the class of injective functions, and $^\perp\{1 \rightarrow 2\} = \mathcal{E}$ is the class of surjective functions. We have $\mathcal{E}^\perp = \mathcal{M}$ and $\mathcal{E} = {}^\perp\mathcal{M}$, and every function factors as $m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$.

We will describe two interesting Freyd-Kelly factorization systems in \mathbf{Gph} (although we will not need them in this paper). Our first system is inspired by Stallings [1983]. We will sketch a complete proof, since it involves a nice use of a “small object argument” (for this idea, see Section 2.1 in Hovey [1999], for instance).

Any pair $\{a', a''\}$ of distinct arcs with a common source node x in a graph X gives a graph morphism $X \rightarrow \underline{X}$, so that $\{a', a''\}$ becomes a single arc a in \underline{X} , and $\{t(a'), t(a'')\}$ becomes a single node y with $t(a) = y$. Stallings [1983] called this type of graph morphism a “folding”.

Note that the word “folding” is used for other types of morphisms in literature on graphs; see the book Hell and Nesetril [2004], for instance. We express the Stallings notion of folding as follows.

Definition: Let \mathbf{V} be the graph with three nodes, 0 and $1'$ and $1''$, and two arcs, a' and a'' , with a' from 0 to $1'$ and a'' from 0 to $1''$. Let $\mathbf{f} : \mathbf{V} \rightarrow \mathbf{A}$ be the graph morphism taking a' and a'' to a . The *elementary folding* associated to a graph morphism $h : \mathbf{V} \rightarrow X$ is the pushout $X \rightarrow \underline{X}$ of the graph morphisms h and \mathbf{f} . A *Folding* is a (possibly transfinite) composition of elementary foldings.

We include some discussion of transfinite composition of graph morphisms within the proof of the next proposition.

The Foldings will form the left class of our first factorization system. The right class is simpler to define.

Definition: A graph morphism $f : X \rightarrow Y$ is *Injecting* if $f : X(x, *) \rightarrow Y(f(x), *)$ is an injective function for all $x \in X_0$.

Proposition 4. Every graph morphism factors as *Injecting* \circ *Folding*.

Proof: Let $f : X \rightarrow Y$ be an arbitrary graph morphism. We will factor f through the composition of a number of steps. We essentially show that every object in \mathbf{Gph} is small, and use a “small object argument” (inspired by Section 2.1 in Hovey [1999]). In fact, if X is infinite we may need a transfinite sequence of steps, so we will index our steps by a well-ordered set, an *ordinal*. We view each ordinal as the set of all smaller ordinals (see Chapter II, Section 3 in Cohen [1966], for instance). Every ordinal α has a *successor*, defined as $\alpha + 1 = \alpha \cup \{\alpha\}$. Let Λ be an ordinal so large that there is no injective function $\Lambda \rightarrow X_1 \times X_1$. We also assume that Λ is not the successor of any ordinal, so that $\lambda \in \Lambda$ implies $\lambda + 1 \in \Lambda$. We view Λ as a category with an object for each element of Λ and one morphism from λ to λ' when $\lambda \leq \lambda'$.

Given $f : X \rightarrow Y$, we will define a functor $X^\bullet : \Lambda \rightarrow \mathbf{Gph}$ equipped with natural transformations f^\bullet and g^\bullet . More precisely, we define X^λ and compatible graph morphisms $f^\lambda : X \rightarrow X^\lambda$ and $g^\lambda : X^\lambda \rightarrow Y$ by transfinite induction, assuming that they are defined for all smaller ordinals. Here each f^λ will be a Folding with $f = g^\lambda \circ f^\lambda$.

For the minimal element $0 \in \Lambda$, we define $X^0 = X$.

If g^λ is not Injecting then we define an elementary folding $X^\lambda \rightarrow X^{\lambda+1}$ by choosing a node x in X^λ having distinct arcs $a', a'' \in X^\lambda(x, *)$ such that $g^\lambda(a') = g^\lambda(a'') \in Y(g^\lambda(x), *)$.

If g^λ is Injecting then we define $X^{\lambda+1} = X^\lambda$ and $f^{\lambda+1} = f^\lambda$ and $g^{\lambda+1} = g^\lambda$.

For λ a limit ordinal (not the successor of any ordinal) we define $X^\lambda = \text{colim}_{\alpha < \lambda} X^\alpha$. The graph morphism f^λ , the colimit of Foldings, is called a *transfinite composition*, and is a Folding, by our definition. Note that if g^λ is Injecting, then we will have $X^\lambda = X^{\lambda'}$ for all $\lambda' > \lambda$, and we may say that the Λ -sequence *stabilizes at* λ . Let us verify that our Λ -sequence stabilizes at some $\lambda \in \Lambda$, so that $f = g^\lambda \circ f^\lambda$ gives our desired factorization.

Each Folding $f^\lambda : X \rightarrow X^\lambda$ determines an equivalence relation $E^\lambda \subseteq X_1 \times X_1$ on the arcs of X . So long as g^λ is not Injecting, we have $E^\lambda \subset E^{\lambda+1}$, a strict inclusion. This shows that the Λ -sequence constructed above eventually stabilizes, since otherwise we could choose a Λ -parametrized family of elements

$p^\lambda \in X_1 \times X_1$ with $p^{\lambda+1} \in E^{\lambda+1} - E^\lambda$. This would give an injective function $\Lambda \rightarrow X_1 \times X_1$, which is impossible by our assumption about the size of Λ . QED

Proposition 5. ((Folding, Injecting)) is a Freyd-Kelly factorization system in Gph.

Proof: We have $\{\mathbf{f}\}^\perp = \text{Injecting}$, from the definitions. Since $\mathbf{f} \in \text{Folding}$, we have $\text{Folding}^\perp \subseteq \text{Injecting}$. For any class of graph morphisms \mathcal{F} , the class ${}^\perp\mathcal{F}$ is closed under pushouts and under transfinite composition. Since \mathbf{f} generates the Foldings under pushouts and transfinite compositions, we have $\text{Folding} \subseteq {}^\perp\text{Injecting}$.

Finally, we use proof by contradiction to show $\text{Injecting}^\perp \subseteq \text{Folding}$. Suppose that $g \notin \text{Folding}$ and $g \in {}^\perp\text{Injecting}$. Factor $g = i \circ f$ with $f \in \text{Folding}$ and $i \in \text{Injecting}$. So f is a graph epimorphism. But the commutative diagram $i \circ f = \text{id}_Y \circ g$ has a (unique) filler h ; then $i \circ h = \text{id}_Y$ shows that h is a graph monomorphism, and $h \circ g = f$ shows that h is a graph epimorphism. So h is a graph isomorphism. But $f = h \circ g$. So g is isomorphic to a Folding, which is a contradiction. QED

We also have a second factorization system, which seems to have interesting connections to algebraic graph theory.

Definition: Let Whisker/Fold denote the class of all graphs morphisms which can be factored as a Whiskering followed by a Folding. Although we will not need it here, we note that Whisker/Fold can also be described as the graph morphisms which can be factored as a Folding followed by a Whiskering.

According to the historical sketch given in Boldi and Vigna [2002], the following basic concept has independently arisen many times in graph theory.

Definition: A graph morphism $f : X \rightarrow Y$ is a *Covering* when $f : X(x, *) \rightarrow Y(f(x), *)$ is a bijective function for all $x \in X_0$.

Other names for this concept (and variants of it) includes *divisor*, *fibration*, *equitable partition*, etc; see Boldi and Vigna [2002].

Coverings would seem to play a fundamental role in algebraic graph theory because of the following.

Fact: If X and Y are finite graphs and $f : X \rightarrow Y$ is a Covering which is surjective on nodes, then the characteristic polynomial of Y divides the characteristic polynomial of X .

Proofs can be found in Chapter 4 of Cvetković, Doob, and Sachs [1978], and in Section 9.3 of Godsil and Royle [2001].

The connection with algebraic graph theory seems to make the following factorization system of graphs especially interesting. This factorization system was the first that we investigated, following some remarks by Steve Schanuel. We state the result without proof here, since we will not need it in this paper:

Proposition 6. ((Whisker/Fold , Covering)) is a Freyd-Kelly factorization system in Gph.

There is a general principle underlying this. Let \mathcal{S} be a suitable category, such as the category of Sets or the category Gph; in particular, \mathcal{S} is to have colimits. Let $i_1 : Y \rightarrow Y + Y$ and $i_2 : Y \rightarrow Y + Y$ be the two canonical morphisms into the coproduct. For any morphism $f : X \rightarrow Y$, let $Y + Y \rightarrow Y +_f Y$ denote the coequalizer of the two morphisms $i_1 \circ f$ and $i_2 \circ f$. Let $\bar{f} : Y +_f Y \rightarrow Y$ denote the induced morphism.

Let S be a set of morphisms in \mathcal{S} . Let $\bar{S} = \{f, \bar{f} : f \in S\}$. Let $\overline{\bar{S}}$ denote the closure of S with respect to the bar operation, all pushouts, and all transfinite compositions. Fajstrup and Rosický [2007] show that in the setting of locally presentable categories, one can guarantee having a factorization system

$$((\overline{\bar{S}}, \overline{\bar{S}}^\dagger)).$$

In our example here, let $S = \{\mathbf{s}\}$. Then \mathbf{f} can be identified with $\bar{\mathbf{s}}$, so that $\bar{S} = \{\mathbf{s}, \mathbf{f}\}$ and $\overline{\bar{S}}^\dagger$ is Coverings. Then $\overline{\bar{S}}$ is all compositions of Whiskerings and Foldings, since it is generated by all transfinite compositions of pushouts of \mathbf{s} and \mathbf{f} (transfinite compositions of pushouts of \mathbf{f} are Foldings, and transfinite compositions of pushouts of \mathbf{s} are Whiskerings).

4. A model structure on the category of graphs.

Suppose that \mathcal{S} is a category with finite limits and finite colimits. We take the following definition from Section 7 of Joyal and Tierney [2006].

Definition: A *model structure* on \mathcal{S} is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of morphisms in \mathcal{S} that satisfies

- 1) “three for two”: if two of the three morphisms $a, b, a \circ b$ belong to \mathcal{W} then so does the third,
- 2) the pair $(\mathcal{C}, \mathcal{F})$ is a weak factorization system (where $\mathcal{C} = \mathcal{C} \cap \mathcal{W}$),
- 3) the pair $(\mathcal{C}, \mathcal{F})$ is a weak factorization system (where $\mathcal{F} = \mathcal{W} \cap \mathcal{F}$).

The morphisms in \mathcal{W} are called *weak equivalences*. The morphisms in \mathcal{C} are called *cofibrations*; and the morphisms in \mathcal{C} are called *acyclic cofibrations*. The morphisms in \mathcal{F} are called *fibrations*, and the morphisms in \mathcal{F} are called *acyclic fibrations*.

Note that, according to Hovey [1999] (page 28), “It tends to be quite difficult to prove that a category admits a model structure. The axioms are always hard to check.”

Recall from Section 2 that the path graph \mathbf{P}_n has nodes $\{0, \dots, n\}$. For $n \geq 0$, the *cycle graph* \mathbf{C}_n is the graph produced by identifying the nodes 0 and n of \mathbf{P}_n . We have $\mathbf{C}_0 = \mathbf{P}_0 = \mathbf{N}$, the graph with one node and no arcs; and \mathbf{C}_1 is the graph with one node, and one arc with source equal to target. Let $C_n(X)$ denote the set of graph morphisms from \mathbf{C}_n to X ; we may call this the set of n -cycles in X .

Definition: A graph morphism $f : X \rightarrow Y$ is *Acyclic* when $C_n(f) : C_n(X) \rightarrow C_n(Y)$ is bijective for all $n > 0$.

Here we exclude $n = 0$, since we don’t want to require that $f_0 : X_0 \rightarrow Y_0$ is a bijection.

The Acyclics contain the Whiskerings, and many other useful graph morphisms.

Morphism classes \mathcal{W} and \mathcal{C} and \mathcal{F} for our Quillen model structure on Gph:

Let \mathcal{W} be the Acyclics. Since Whiskerings are Acyclic, we may let \mathcal{C} be the Whiskerings. It follows that \mathcal{F} must be the Surjectings. Then \mathcal{F} must be the Acyclic Surjectings. Finally, we define \mathcal{C} to be ${}^\dagger \mathcal{F}$. In the next few propositions we show directly that this does indeed define a model structure on Gph.

Proposition 7. The Acyclics satisfy the “three for two” property.

Proof: This is easy, since Acyclics are defined functorially. Consider $h = f \circ g$. Then $C_n(h) = C_n(f) \circ C_n(g)$ for all n . But the class of bijective functions in the category of sets satisfies the “three for two” property. QED.

We have already shown in Section 2 that $(\mathcal{C}, \mathcal{F})$ is a weak factorization system. It remains to do the same for $(\mathcal{C}, \mathcal{F})$. We use the following facts, which are easy to verify directly from the definition $\mathcal{C} = {}^\dagger \mathcal{F}$.

- 0) Every composition of graph morphisms in \mathcal{C} is in \mathcal{C} .
- 1) Every Whiskering is in \mathcal{C} .
- 2) For any set I and $n > 0$, if $A_i \rightarrow B_i$ is in \mathcal{C} for all $i \in I$, then $\sum_{i \in I} A_i \rightarrow \sum_{i \in I} B_i$ is in \mathcal{C} .
- 3) For any set I and $n > 0$, the graph morphism $\emptyset \rightarrow I \times C_n$ is in \mathcal{C} .
- 4) For any set I , the graph morphism $I \times C_n \rightarrow C_n$ is in \mathcal{C} .

In 3) and 4) we view set I as a discrete graph and we view graph $I \times C_n$ as a summand of copies of C_n .

Proposition 8. Every graph morphism g factors as $g = f \circ c$ with $c \in \mathcal{C}$ and $f \in \mathcal{F}$.

Proof: Given any graph morphism $g : X \rightarrow Y$, we factor g in three steps.

First, we let C be the disjoint union of a copy of \mathbf{C}_n for each element of $C_n(Y)$ which is not the image of $C_n(g) : C_n(X) \rightarrow C_n(Y)$. Let $h : C \rightarrow Y$ be the graph morphism which sends each summand cycle of C to its image in Y . Let $X' = X + C$, and let $g' : X \rightarrow X'$ denote the inclusion $X \rightarrow X + C$, and let $f' : X' \rightarrow Y$ denote the graph morphism $X + C \rightarrow Y$ determined by $g : X \rightarrow Y$ and $h : C \rightarrow Y$. Then $g = f' \circ g'$, and $g' \in \mathcal{C}$, and $C_n(f') : C_n(X') \rightarrow C_n(Y)$ is surjective for all $n > 0$.

Next, we let $J = \{(c, n) : c \in C_n(Y)\}$, with $j : \sum_J I_c \times \mathbf{C}_n \rightarrow \sum_J \mathbf{C}_n$ where I_c is the preimage of c for the function $C_n(f') : C_n(X') \rightarrow C_n(Y)$. Also let $k : \sum_J I_c \times \mathbf{C}_n \rightarrow X'$ be the graph morphism which sends each summand cycle to the corresponding cycle in X' , and let $\ell : \sum_J \mathbf{C}_n \rightarrow Y$ be the graph morphism which sends each summand cycle to the corresponding cycle in Y . Let $g'' : X' \rightarrow X''$ denote the pushout of j along k . Let $f'' : X'' \rightarrow Y$ be the pushout graph morphism induced by ℓ and f' . Then $g'' \in \mathcal{C}$ and $f' = f'' \circ g''$, and $C_n(f'') : C_n(X'') \rightarrow C_n(Y)$ is bijective for all $n > 0$, so that $f'' \in \mathcal{W}$.

Finally, we factor $f'' = f''' \circ g'''$ with Whiskering $g''' : X'' \rightarrow X'''$ and Surjecting $f''' : X''' \rightarrow Y$, as in Section 2. Then $g''' \in \mathcal{C}$ and $f''' \in \mathcal{W} \cap \mathcal{F}$.

Thus, $g = c \circ f$ with $c = g''' \circ g'' \circ g'$ in \mathcal{C} , and $f = f'''$ in \mathcal{F} . QED

Proposition 9. $(\mathcal{C}, \mathcal{F})$ is a weak factorization system in Gph.

Proof: We have $\mathcal{C} = {}^\dagger\mathcal{F}$, by definition. This shows also that $\mathcal{F} \subseteq \mathcal{C}^\dagger$. It remains only to show that $\mathcal{C}^\dagger \subseteq \mathcal{F}$. But this is easy. Consider the Whiskering $\mathbf{s} : \mathbf{N} \rightarrow \mathbf{A}$ from Section 2, and the graph morphisms $\mathbf{i}_n : 0 \rightarrow \mathbf{C}_n$, $\mathbf{j}_n : \mathbf{C}_n + \mathbf{C}_n \rightarrow \mathbf{C}_n$ as in 3) and 4) above. These are all in \mathcal{C} , since each can be lifted against any graph morphism in \mathcal{F} . But if $g \notin \mathcal{F}$ then we can show failure of lifting for either \mathbf{s} or some \mathbf{i}_n or \mathbf{j}_n . QED

Corollary: Our morphism classes \mathcal{W} and \mathcal{C} and \mathcal{F} provide a Quillen model structure for the category Gph.

The above proofs show how the graph morphisms $\mathbf{s} : \mathbf{N} \rightarrow \mathbf{A}$ together with $\mathbf{i}_n : 0 \rightarrow \mathbf{C}_n$ and $\mathbf{j}_n : \mathbf{C}_n + \mathbf{C}_n \rightarrow \mathbf{C}_n$, for $n > 0$, generate our class \mathcal{C} of Cofibrations. This situation is a special case of a general principle in pre-sheaf categories; see Proposition 7.5 in Joyal and Tierney [2006], for instance.

5. Zeta series and almost isospectral graphs.

Ihara zeta functions of graphs are usually discussed in a setting of “unoriented” or “symmetric” graphs; see Kotani and Sunada [2000], for instance. We need a version suitable for *directed* graphs (in this section we may refer to objects of our category Gph as directed graphs, for emphasis). There is a nice treatment of zeta series of finite directed graphs in Section 2 of Kotani and Sunada [2000]; we will follow them here, but with our own terminology.

Definition: A *finite graph* is one with finitely many nodes and arcs. The *zeta series* of a finite directed graph X is the formal power series

$$Z(u) = \exp\left(\sum_{m=1}^{\infty} c_m \frac{u^m}{m}\right),$$

where $c_m = |C_m(X)|$ for $m > 0$.

See the appendix for some motivation for this definition, including how it relates to an Euler product expansion in terms of “primes”.

Example: if X is the graph with one node and n arcs, then $c_m = n^m$ and

$$\sum_{m=1}^{\infty} c_m \frac{u^m}{m} = \sum_{m=1}^{\infty} \frac{n^m u^m}{m} = -\log(1 - nu) \quad \text{so that} \quad Z(u) = \frac{1}{1 - nu}.$$

Definition: Let X be a finite graph. Let $\mathbf{R}X_0$ denote the real vector space with basis the nodes of X . The *adjacency operator* A for X is the linear transformation $A : \mathbf{R}X_0 \rightarrow \mathbf{R}X_0$ determined by

$$A(x) = \sum_{a \in X(x,*)} t(a)$$

for $x \in X_0$. The *characteristic polynomial* of X is defined as $a(x) = \det(xI - A)$, the characteristic polynomial of the adjacency operator A for X . If X has n nodes, then $a(x)$ is a monic polynomial of degree n , and the *reversed characteristic polynomial* of X is defined to be $u^n a(u^{-1})$.

If we totally order the nodes of X , then the adjacency operator is represented by the square matrix A with entry $A_{j,i}$ equal to the number of arcs in X from the i^{th} node to the j^{th} node.

Note that the reversed characteristic polynomial of a finite graph X has constant term 1, and is thus a unit in the ring of formal power series with integer coefficients.

Proposition 10. If X is a finite graph with n nodes then the zeta series of X satisfies

$$Z(u) = \det(I - uA)^{-1} = \frac{1}{u^n a(u^{-1})}.$$

Proof: Let A be any endomorphism of an n -dimensional real vector space V . We have $\det(I - uA) = u^n \det(u^{-1}I - A)$, which proves the second equality. One can check the first equality by induction on the dimension of V , since both sides are multiplicative for short exact sequences of vector spaces endowed with

endomorphisms. Or, when V has a basis of eigenvectors for A with eigenvalues $\lambda_1, \dots, \lambda_n$, we can see the first equality from $-\log(1-x) = \sum_k \frac{x^k}{k}$ and

$$\exp\left(\sum_{m=1}^{\infty} \sum_{i=1}^n \lambda_i^m \frac{u^m}{m}\right) = \prod_{i=1}^n \exp(-\log(1-\lambda_i u)) = \prod_{i=1}^n \frac{1}{1-\lambda_i u} = \det(I - uA)^{-1}.$$

QED

Example (continued): if X is the graph with one node and n arcs then $a(x) = x-n$ and $u^1 a(u^{-1}) = 1-nu$, which agrees with $Z(u) = \frac{1}{1-nu}$.

Proposition 11. If X and Y are finite graphs and $f : X \rightarrow Y$ is an acyclic morphism then $Z_X = Z_Y$.

Proof: This is clear from the definition, since $|C_m(X)| = |C_m(Y)|$ for all $m > 0$. QED

Definition: The eigenvalues of the adjacency operator for X may be called the *spectrum* of X (even though the adjacency operator is not necessarily a diagonalizable operator). We say that two finite graphs X and Y are *isospectral* if they have the same characteristic polynomial. We say that X and Y are *almost isospectral* if they have the same reversed characteristic polynomial.

Loosely speaking, X and Y are almost isospectral if and only if they have the same non-zero eigenvalues.

Corollary: If X and Y are finite graphs with $Z_X = Z_Y$ then X and Y are almost isospectral.

Proof: This follows immediately from the preceding two propositions. QED

An appendix on the zeta series and its Euler product expansion.

Here is a little history, taken from Thomas [1977], of how our zeta series for finite directed graphs relates to the famous zeta functions from number theory.

The zeta function of Euler and Riemann. Let p range over the prime numbers. Then

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

Dedekind's zeta function for algebraic number fields. Let A be the ring of integers in an algebraic number field K (so K is a finite extension over the field \mathbb{Q} of rational numbers). Let $N(I) = |A/I|$ for any non-zero ideal in A . Then

$$\zeta(s) = \sum_I \frac{1}{N(I)^s} = \prod_P (1 - N(P)^{-s})^{-1}$$

where I ranges over the non-zero principal ideals in A and P ranges over the prime ideals in A .

A zeta function for algebraic function fields. Let A is the ring of integers in an algebraic function field (so K is a finite extension over the field $\mathbb{F}_q(x)$ of rational functions with coefficients in the field \mathbb{F}_q with q elements). But here $N(I) = |A/I| = q^{\nu(I)}$ for any non-zero ideal in A , where $\nu(I)$ is the dimension of A/I as a finite dimensional vector space over \mathbb{F}_q , so that

$$\zeta(s) = Z(u)|_{u=q^{-s}} \quad \text{for} \quad Z(u) = \prod_P (1 - u^{\nu(P)})^{-1}.$$

The zeta function for a projective variety over a finite field from Weil [1949] is a completed version of this.

The form of zeta series that we use in section 5 seems ultimately based on the following observation. For a square matrix of a fixed size, the knowledge of the trace of A^n for all n is equivalent to the knowledge of the characteristic polynomial of A , in that

$$\det(1 - uA)^{-1} = \exp\left(\sum_{n=1}^{\infty} \text{Trace}(A^n) \frac{u^n}{n}\right).$$

The proof is just like that of the comparable proposition in section 5.

If A is a permutation matrix then the trace of A^n counts the number of fixed points of A^n . This is one of the ideas behind the Lefschetz fixed point theorem, the Weil zeta function used in the Weil conjectures, and other dynamical zeta functions such as the Selberg zeta function in Riemannian geometry and the Ihara zeta function (see Ruelle [2002] for instance).

In our setting of graphs, we merely use that if A is the adjacency matrix of a finite graph X , then the trace of A^n counts the number of cycles of length n in X . This leads to the following Euler product expansion, analogous to the one for algebraic function fields.

The cycle graph \mathbf{C}_n has nodes i for $0 \leq i < n$. If m divides n then we have a graph morphism $\pi : \mathbf{C}_n \rightarrow \mathbf{C}_m$ given by sending node i to node $i \bmod m$.

Definition: A cycle $c : \mathbf{C}_{km} \rightarrow X$ is a k -multiple if $c = c' \circ \pi$ for some cycle $c' : \mathbf{C}_m \rightarrow X$. A *prime cycle of length n in X* is a cycle $c : \mathbf{C}_n \rightarrow X$ which is not a k -multiple for any $k > 1$. Let us say that two cycles $c, c' : \mathbf{C}_n \rightarrow X$ are *shift equivalent* if $c' = c \circ \tau^i$ for some i , where $\tau^i : \mathbf{C}_n \rightarrow \mathbf{C}_n$ is the *shift morphism* sending node j to node $j + i \bmod n$. Let us say that a *prime P in X* is an equivalence class of prime cycles in X , and that $\nu(P)$ is the length of the prime P .

This makes sense, since shift equivalence is an equivalence relation on $C_n(X)$, and on prime cycles of length n in X .

Proposition 12. The Euler product expansion for the zeta function of a finite graph is given by

$$Z(u) = \prod_P (1 - u^{\nu(P)})^{-1}$$

where P ranges over all primes in X and $\nu(P)$ is the length of P .

Proof: Let \bar{c}_k be the number of primes P of length k . Then $c_m = \sum_{\{k:k|m\}} k \bar{c}_k$ and we have

$$\log \prod_P (1 - u^{\nu(P)})^{-1} = \sum_P \sum_{k=1}^{\infty} \frac{u^{k|P|}}{k} = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \sum_{\nu(P)=\ell} \frac{u^{k\ell}}{k} = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \bar{c}_\ell \frac{u^{k\ell}}{k} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\ell|m} \ell \bar{c}_\ell u^m = \sum_{m=1}^{\infty} c_m \frac{u^m}{m},$$

which is $\log Z(u)$. Here equality one and equality four follow from

$$-\log(1 - u^v) = \sum_k \frac{u^{kv}}{k} \quad \text{and} \quad m = k\ell \iff \frac{\ell}{m} = \frac{1}{k}.$$

QED

Example (continued): for X the graph with one node and two arcs, we must have

$$\frac{1}{1-2u} = (1-u)^{-2}(1-u^2)^{-1}(1-u^3)^{-2} \dots = \prod (1-u^k)^{\bar{c}_k}.$$

This is related to the cyclotomic (necklace) identity; see Dress and Siebeneicher [1989], for instance.

An appendix on the Galois connection of a binary relation:

Suppose we are given a binary relation $\bowtie \subseteq \mathcal{L} \times \mathcal{R}$. We write $\ell \bowtie r$ to mean $(\ell, r) \in \bowtie$. Birkhoff [1940] and Ore [1944] developed the *Galois connection* formalism that follows.

For $L \subseteq \mathcal{L}$ and $R \subseteq \mathcal{R}$ we define

$$L^* = \{r \in \mathcal{R} : \forall \ell \in L, \ell \bowtie r\} \quad \text{and, dually,} \quad {}^*R = \{\ell \in \mathcal{L} : \forall r \in R, \ell \bowtie r\}.$$

We define $\bullet L = {}^*(L^*)$ and say that L is *left-closed* iff $L = \bullet L$, and dually on the right.

The following observations are easy to verify:

- 1) $L_1 \subseteq L_2$ implies $L_2^* \subseteq L_1^*$, and dually on the right.

- 2) $L \subseteq \bullet L$, and dually on the right.
- 3) $L_1 \subseteq L_2$ implies $\bullet L_1 \subseteq \bullet L_2$, and dually on the right.
- 4) $\bullet\bullet L = \bullet L$, and dually on the right.
- 5) $L^* = (\bullet L)^*$, and dually on the right.
- 6) $*R$ is left-closed for any R , and dually on the right.
- 7) $*R_1 = *R_2$ if and only if $R_1 = R_2$, and dually on the right.
- 8) every left-closed is of the form $*R$; and dually on the right.
- 9) the intersection of any collection of left-closed is left-closed, and dually on the right.

The above ideas can also be viewed as a basic (contravariant) example of adjoint functors; see Chapter IV Section 5 in Mac Lane [1971].

References:

- [1940] G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloquium Publ., 25, New York, 1940.
- [2002] P. Boldi and S. Vigna, Fibrations of graphs, Discrete Math., 243 (2002), 21-66.
- [2002] D-C. Cisinski, Théories homotopiques dans les topos, Journal of Pure and Applied Algebra, 174 (2002), 43-82.
- [1966] P. J. Cohen, Set Theory and the Continuum Hypothesis, W.A. Benjamin, NY, 1966.
- [1978] D. M. Cvetković, M. Doob, and H. Sachs, Spectra of Graphs, Academic Press, 1978.
- [1983] A. Dold, Fixed point indices of iterated maps, Invent. math., 74 (1983), 419-435.
- [1989] A. W. M. Dress and C. Siebeneicher, The Burnside ring of the infinite cyclic group and its relations to the necklace algebra, λ -rings, and the universal ring of witt vectors. Adv. in Math., 78 (1989), 1-41.
- [1995] W.G. Dwyer and J. Spalinski, Homotopy theories and model categories, in Handbook of Algebraic Topology, Editor I. M. James, 73-126, Elsevier, 1995.
- [1999] E. E. Enochs and I. Herzog, A homotopy of quiver morphisms with applications to representations, Canad. J. Math., 51(2) (1999), 294-308.
- [2007] L. Fajstrup and J. Rosický, A convenient category for directed homotopy, arXiv:math/07/08/3937v1
- [1972] P. J. Freyd and G. M. Kelly, Categories of continuous functors, I, J. of Pure and Applied Algebra, 2 (1972), 169-191.
- [2001] C. Godsil and G. Royle, Algebraic Graph Theory, Springer-Verlag, New York (2001).
- [2004] P. Hell and J. Nešetřil, Graphs and Homomorphisms, vol. 28 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2004.
- [1999] M. Hovey, Model Categories, Amer. Math. Soc., Providence, 1999.
- [1991] A. Joyal and M. Tierney, Strong stacks and classifying spaces, p213-236 in Category Theory (Como, 1990), Lecture Notes in Math., 1488, Springer, Berlin, 1991.
- [2006] A. Joyal and M. Tierney, Quasi-categories vs Segal spaces, arXiv:math/06/07/820v1
- [2000] M. Kotani and T. Sunada, Zeta functions of finite graphs, J. Math. Sci. Univ. Tokyo, 7 (2000), 7-25.
- [1989] F. W. Lawvere, Qualitative distinctions between some toposes of generalized graphs, In Categories in computer science and logic (Boulder 1987), vol. 92 of Contemp. Math., 261-299, Amer. Math. Soc., Providence, 1989.
- [1997] F. W. Lawvere and S. H. Schanuel, Conceptual Mathematics: a first introduction to categories, Cambridge University Press, Cambridge, 1997.
- [1971] Mac Lane, Categories for the Working mathematician, Graduate Texts in Mathematics, Springer-Verlag, New York, 1971.
- [1994] S. Mac Lane and I. Moerdijk. Sheaves in Geometry and Logic: a first introduction to topos theory, Universitext, Springer-Verlag, New York (1994)
- [1999] F. Morel and V. Voevodsky, A^1 -homotopy of schemes, Inst. Hautes Études Sci., Pub. Math. 90 (1999), 45-143.

- [1944] O. Ore, Galois connexions, Trans. Amer. Math. Soc., 55 (1944), 493-513.
- [1967] D. G. Quillen, Homotopical Algebra, Lecture Notes in Mathematics No. 43, Springer-Verlag, Berlin, 1967.
- [2002] D. Ruelle, Dynamical zeta functions and transfer operators, Notices of the Amer. Math. Soc., 49 no. 8 (2002), 887-895.
- [1983] J. R. Stallings, Topology of finite graphs, Invent. Math., 71 (1983), 551-565.
- [1977] A. D. Thomas, Zeta-functions: an introduction to algebraic geometry, Pitman, London, 1977.
- [1980] R. W. Thomason, Cat as a closed model category, Cahiers Topologie Géom. Différentielle, 21 no. 3 (1980), 305-324.
- [1949] A. Weil, Number of solutions of equations in finite fields, Bull. Am. Math Soc., 55 (1949), 497-508.

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